

**CASCADE CONNECTIONS OF LINEAR SYSTEMS AND
FACTORIZATIONS OF HOLOMORPHIC OPERATOR FUNCTIONS
AROUND A MULTIPLE ZERO IN SEVERAL VARIABLES**

Dmitriy S. Kalyuzhniy

We show that the factorization problem $\theta(z) = \theta_2(z)\theta_1(z)$ is solvable in the class of Hilbert space operator-valued functions holomorphic on some neighbourhood of $z = 0$ in \mathbb{C}^N and having a zero at $z = 0$ (here $\theta(z)$ has a multiple zero at $z = 0$). Such a factorization problem becomes more complicated if we demand for $\theta(z)$, $\theta_1(z)$ and $\theta_2(z)$ to be Agler–Schur-class functions on the polydisk \mathbb{D}^N and for the factorization identity to hold in \mathbb{D}^N . In this case we reduce it to the problem on the existence of a cascade decomposition for certain multiparametric linear system α —a conservative realization of $\theta(z)$, and give the criterion for its solvability in terms of common invariant subspaces for the N -tuple of main operators of α .

1 Preliminaries

In this section we recall the necessary information on multiparametric linear systems, the Agler–Schur class of operator-valued functions on the open unit polydisk \mathbb{D}^N , and related realization theorems for holomorphic operator-valued functions of several complex variables.

In [7] we have introduced *multiparametric discrete time-invariant linear dynamical systems* of the form

$$\alpha : \begin{cases} x(t) = \sum_{k=1}^N (A_k x(t - e_k) + B_k u(t - e_k)), \\ y(t) = \sum_{k=1}^N (C_k x(t - e_k) + D_k u(t - e_k)), \end{cases} \quad (t \in \mathbb{Z}^N, |t| > 0) \quad (1.1)$$

where $|t| := \sum_{k=1}^N t_k$, $e_k := (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}^N$ (here 1 is on the k -th place and zeros are otherwise); for all $t \in \mathbb{Z}^N$ such that $|t| \geq 0$ $x(t)$ ($\in \mathcal{X}$), $u(t)$ ($\in \mathcal{U}$), and for all $t \in \mathbb{Z}^N$ such that $|t| > 0$ $y(t)$ ($\in \mathcal{Y}$) are respectively *states*, *input data* and *output data* of α , and $\mathcal{X}, \mathcal{U}, \mathcal{Y}$ are separable Hilbert spaces; for all $k \in \{1, \dots, N\}$ A_k, B_k, C_k, D_k are bounded linear operators acting in corresponding pairs of Hilbert spaces. If one denotes an N -tuple of operators T_k ($k = 1, \dots, N$) by $\mathbf{T} := (T_1, \dots, T_N)$ then for system (1.1) one may use the short notation $\alpha =$

$(N; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$. If $z = (z_1, \dots, z_N) \in \mathbb{C}^N$ then we set $z\mathbf{T} := \sum_{k=1}^N z_k T_k$. The operator-valued function

$$\theta_\alpha(z) = z\mathbf{D} + z\mathbf{C}(I_{\mathcal{X}} - z\mathbf{A})^{-1}z\mathbf{B} \quad (1.2)$$

(here $I_{\mathcal{X}}$ is the identity operator on \mathcal{X}) holomorphic on some neighbourhood of $z = 0$ in \mathbb{C}^N is called the *transfer function* of a system α of the form (1.1). We have the following result [6]: an arbitrary function $\theta(z)$ which is holomorphic on some neighbourhood Γ of $z = 0$ in \mathbb{C}^N and vanishing at $z = 0$, whose values are from $[\mathcal{U}, \mathcal{Y}]$ (we use the notation $[\mathcal{U}, \mathcal{Y}]$ for the Banach space of all bounded linear operators mapping a separable Hilbert space \mathcal{U} into a separable Hilbert space \mathcal{Y}) can be realized as the transfer function of some system $\alpha = (N; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$, i.e. $\theta(z) = \theta_\alpha(z)$ in some neighbourhood (possibly, smaller than Γ) of $z = 0$. We call $\alpha = (N; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ a *dissipative* (resp., *conservative*) *scattering system* if for any $\zeta \in \mathbb{T}^N$ (the N -fold unit torus)

$$\zeta \mathbf{G}_\alpha := \begin{bmatrix} \zeta \mathbf{A} & \zeta \mathbf{B} \\ \zeta \mathbf{C} & \zeta \mathbf{D} \end{bmatrix} \in [\mathcal{X} \oplus \mathcal{U}, \mathcal{X} \oplus \mathcal{Y}]$$

is a contractive (resp., unitary) operator. Recall that the *Agler–Schur class* $S_N(\mathcal{U}, \mathcal{Y})$ (see [1]) consists of all holomorphic functions $\theta(z) = \sum_{t \in \mathbb{Z}_+^N} \widehat{\theta}_t z^t$ on \mathbb{D}^N with values in $[\mathcal{U}, \mathcal{Y}]$ (here $\mathbb{Z}_+^N := \{t \in \mathbb{Z}^N : t_k \geq 0, k = 1, \dots, N\}$, $z^t := \prod_{k=1}^N z_k^{t_k}$) such that for any separable Hilbert space \mathcal{H} , any N -tuple $\mathbf{T} = (T_1, \dots, T_N)$ of commuting contractions on \mathcal{H} and any positive $r < 1$ one has

$$\|\theta(r\mathbf{T})\| \leq 1 \quad (1.3)$$

where

$$\theta(r\mathbf{T}) = \theta(rT_1, \dots, rT_N) := \sum_{t \in \mathbb{Z}_+^N} \widehat{\theta}_t \otimes (r\mathbf{T})^t \quad (1.4)$$

(the convergence of this series is understood in the sense of norm in $[\mathcal{U} \otimes \mathcal{H}, \mathcal{Y} \otimes \mathcal{H}]$). For $N = 1$ due to the von Neumann inequality [9] we have $S_N(\mathcal{U}, \mathcal{Y}) = S(\mathcal{U}, \mathcal{Y})$, i.e. the Schur class consisting of all functions holomorphic on the open unit disk \mathbb{D} with contractive values from $[\mathcal{U}, \mathcal{Y}]$. In [7] we have proved that the class of transfer functions of N -parametric conservative scattering systems with the input space \mathcal{U} and the output space \mathcal{Y} coincides with the subclass $S_N^0(\mathcal{U}, \mathcal{Y})$ of $S_N(\mathcal{U}, \mathcal{Y})$ consisting of functions vanishing at $z = 0$. Moreover, the conservative realization $\alpha = (N; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ of an arbitrary function $\theta(z) \in S_N^0(\mathcal{U}, \mathcal{Y})$ can be chosen *closely connected*, i.e. such that $\mathcal{X}_{cc} = \mathcal{X}$ where

$$\mathcal{X}_{cc} := \bigvee_{p, k, j} p(\mathbf{A}, \mathbf{A}^*) (B_k \mathcal{U} + C_j^* \mathcal{Y}) \quad (1.5)$$

(here “ $\bigvee_\nu \mathcal{L}_\nu$ ” denotes the closure of the linear span of subsets \mathcal{L}_ν in \mathcal{X} , p runs over the set of all monomials in $2N$ non-commuting variables, k and j run over the set $\{1, \dots, N\}$). Notice that the close connectedness of a conservative scattering system $\alpha = (N; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is equivalent to the condition that the linear pencil $\zeta \mathbf{A}$ ($\zeta \in \mathbb{T}^N$) of contractive operators is completely non-unitary, i.e. there is no proper subspace in \mathcal{X} reducing $\zeta \mathbf{A}$ to a unitary operator for each $\zeta \in \mathbb{T}^N$.

2 Certain factorization problems in several complex variables

In this section we study factorizations of operator-valued functions which are holomorphic on a neighbourhood of some point $z = z_0$ in \mathbb{C}^N and have a zero at $z = z_0$. Without loss of generality, one can consider $z_0 = 0$. For this case we say that such a function $\theta(z)$, which is not vanishing identically on a neighbourhood of $z = 0$, has a zero of *multiplicity* $m = m(\theta)$ at $z = 0$ if m is the least number of a non-zero term in the expansion of $\theta(z)$ in homogeneous polynomials (e.g., see [8]).

Problem 2.1 Given a function $\theta(z)$ which is holomorphic on the neighbourhood Γ of $z = 0$ in \mathbb{C}^N , takes values from $[\mathcal{U}, \mathcal{Y}]$, and has a zero of multiplicity $m(\theta) > 1$ at $z = 0$, find a separable Hilbert space \mathcal{V} and functions $\theta_1(z)$, $\theta_2(z)$ which are holomorphic on some neighbourhoods of $z = 0$, take values from $[\mathcal{U}, \mathcal{V}]$ and $[\mathcal{V}, \mathcal{Y}]$ respectively, $\theta_1(0) = 0$, $\theta_2(0) = 0$, and

$$\theta(z) = \theta_2(z)\theta_1(z)$$

holds in some neighbourhood (possibly, smaller than Γ) of $z = 0$.

This problem is solvable; in fact, the following more strong statement is true.

Theorem 2.2 Let $\theta(z)$ be a function holomorphic on some neighbourhood Γ of $z = 0$ in \mathbb{C}^N , taking values from $[\mathcal{U}, \mathcal{Y}]$ and having a zero of multiplicity $m = m(\theta) > 0$ at $z = 0$. Then there exist separable Hilbert spaces $\mathcal{Y}^{(0)} = \mathcal{Y}, \mathcal{Y}^{(1)}, \dots, \mathcal{Y}^{(m)}$, operators $L_k^{(j)} \in [\mathcal{Y}^{(j)}, \mathcal{Y}^{(j-1)}]$ ($j = 1, \dots, m$; $k = 1, \dots, N$), and a function $\phi(z)$ which is holomorphic on some neighbourhood of $z = 0$, takes values from $[\mathcal{U}, \mathcal{Y}^{(m)}]$, and $\phi(0) \neq 0$, such that

$$\theta(z) = z \mathbf{L}^{(1)} \dots z \mathbf{L}^{(m)} \phi(z) \tag{2.1}$$

holds in some neighbourhood (possibly, smaller than Γ) of $z = 0$.

PROOF. As it was said in Section 1, we have $\theta(z) = \theta_\alpha(z)$ in some neighbourhood of $z = 0$ where $\alpha = (N; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is some N -parametric system. If $m = 1$ then setting $\mathcal{Y}^{(1)} := \mathcal{X} \oplus \mathcal{U}$, $L_k^{(1)} := [C_k^{(1)} \ D_k^{(1)}]$ ($k = 1, \dots, N$), $\phi(z) :=$

$\begin{bmatrix} (I_{\mathcal{X}} - z\mathbf{A})^{-1}z\mathbf{B} \\ I_{\mathcal{U}} \end{bmatrix}$, we get from (1.2) the equality $\theta(z) = z\mathbf{L}^{(1)}\phi(z)$, with $\phi(0) \neq 0$, i.e. (2.1) is true. Let us apply the induction on m . Suppose that the statement is true for $m - 1$ ($m > 1$). Since $m = m(\theta) > 1$ implies $\mathbf{D} = (0, \dots, 0)$ and (1.2) turns into $\theta(z) = z\mathbf{C}(I_{\mathcal{X}} - z\mathbf{A})^{-1}z\mathbf{B}$ we can set $\mathcal{Y}^{(1)} := \mathcal{X}$, $L_k^{(1)} := C_k$ ($k = 1, \dots, N$), $\tilde{\theta}(z) := (z\mathbf{A})^{m-2}(I_{\mathcal{X}} - z\mathbf{A})^{-1}z\mathbf{B}$ and get $\theta(z) = z\mathbf{L}^{(1)}\tilde{\theta}(z)$. Indeed, $z\mathbf{L}^{(1)}\tilde{\theta}(z) = z\mathbf{C}(z\mathbf{A})^{m-2}(I_{\mathcal{X}} - z\mathbf{A})^{-1}z\mathbf{B} = z\mathbf{C}(I_{\mathcal{X}} - z\mathbf{A})^{-1}z\mathbf{B} = \theta(z)$ since $z\mathbf{C}(z\mathbf{A})^j z\mathbf{B} = 0$ identically for $j < m - 2$ (the case $m = 2$ is obvious). It is clear that $m(\tilde{\theta}) = m - 1$. By the supposition of the induction there exist separable Hilbert spaces $\mathcal{Y}^{(2)}, \dots, \mathcal{Y}^{(m)}$, operators $L_k^{(j)} \in [\mathcal{Y}^{(j)}, \mathcal{Y}^{(j-1)}]$ ($j = 2, \dots, m$; $k = 1, \dots, N$), and a holomorphic function $\phi(z)$ with values from $[\mathcal{U}, \mathcal{Y}^{(m)}]$ such that $\phi(0) \neq 0$ and $\tilde{\theta}(z) = z\mathbf{L}^{(2)} \dots z\mathbf{L}^{(m)}\phi(z)$ in some neighbourhood of $z = 0$. Then (2.1) is true, and the proof is complete. \square

Corollary 2.3 *If $\theta(z)$ is a homogeneous polynomial of degree m then in the statement of Theorem 2.2 one can choose $\mathcal{Y}^{(m)} = \mathcal{U}$, $\phi(z) = I_{\mathcal{U}}$, and (2.1) turns into*

$$\theta(z) = z\mathbf{L}^{(1)} \dots z\mathbf{L}^{(m)}. \quad (2.2)$$

If $m > 1$ then, moreover,

$$\theta(z) = z\mathbf{C}(z\mathbf{A})^{m-2}z\mathbf{B}. \quad (2.3)$$

PROOF. The case $m = 1$ is trivial. For $m > 1$ we obtain (2.3) from (1.2) by virtue of the uniqueness of Maclaurin's expansion for $\theta(z)$. \square

Similarly, one can obtain the right-hand analogue of Theorem 2.2.

Theorem 2.4 *Let $\theta(z)$ be a function holomorphic on some neighbourhood Γ of $z = 0$ in \mathbb{C}^N , taking values from $[\mathcal{U}, \mathcal{Y}]$ and having a zero of multiplicity $m = m(\theta) > 0$ at $z = 0$. Then there exist separable Hilbert spaces $\mathcal{U}^{(0)} = \mathcal{U}, \mathcal{U}^{(1)}, \dots, \mathcal{U}^{(m)}$, operators $R_k^{(j)} \in [\mathcal{U}^{(j-1)}, \mathcal{U}^{(j)}]$ ($j = 1, \dots, m$; $k = 1, \dots, N$), and a function $\psi(z)$ which is holomorphic on some neighbourhood of $z = 0$, takes values from $[\mathcal{U}^{(m)}, \mathcal{Y}]$, and $\psi(0) \neq 0$, such that*

$$\theta(z) = \psi(z)z\mathbf{R}^{(m)} \dots z\mathbf{R}^{(1)} \quad (2.4)$$

holds in some neighbourhood (possibly, smaller than Γ) of $z = 0$.

Let us remark that Theorems 2.2 and 2.4 are multivariate generalizations of the theorem on a multiple zero for functions of one complex variable which are different, even for the scalar-valued case, from the celebrated Weierstrass Preparation Theorem (WPT). We have in (2.1) and (2.4) the products of linear factors (i.e., the homogeneous polynomials) instead of the Weierstrass polynomial in WPT which is, in fact, a polynomial in one distinguished variable and not necessarily a polynomial

in other variables (see, e.g., [8]). However, these linear factors can be operator-valued, and, in contrast to WPT, factorizations (2.1) and (2.4) are non-unique. Let us remark also that for $N = 1$ factorizations (2.2) and (2.3) are trivial and reduced to $\theta(z) = z^m L$ where $L \in [\mathcal{U}, \mathcal{Y}]$.

Problem 2.5 Given a function $\theta(z) \in S_N^0(\mathcal{U}, \mathcal{Y})$ such that $m(\theta) > 1$, find a separable Hilbert space \mathcal{V} and functions $\theta_1(z) \in S_N^0(\mathcal{U}, \mathcal{V})$, $\theta_2(z) \in S_N^0(\mathcal{V}, \mathcal{Y})$ such that

$$\theta(z) = \theta_2(z)\theta_1(z). \quad (z \in \mathbb{D}^N) \quad (2.5)$$

For the special case of a homogeneous polynomial Problem 2.5 is solvable.

Theorem 2.6 *If $\theta(z) \in S_N^0(\mathcal{U}, \mathcal{Y})$ is a homogeneous polynomial of degree m then (2.2) holds for $z \in \mathbb{D}^N$ with linear factors $z\mathbf{L}^{(j)} \in S_N^0(\mathcal{Y}^{(j)}, \mathcal{Y}^{(j-1)})$ ($j = 1, \dots, m$) (here $\mathcal{Y}^{(0)} = \mathcal{Y}$ and $\mathcal{Y}^{(m)} = \mathcal{U}$).*

PROOF. As it was said in Section 1 there exists a conservative scattering system $\alpha = (N; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ such that $\theta(z) = \theta_\alpha(z)$ in \mathbb{D}^N . If $m = 1$ then $\theta(z) = \theta_\alpha(z) = z\mathbf{D}$, thus the statement is valid with $\mathbf{L}^{(1)} = \mathbf{D}$. If $m > 1$ then $\mathbf{D} = (0, \dots, 0)$ and (2.3) holds. Let us show that $z\mathbf{B} \in S_N^0(\mathcal{U}, \mathcal{X})$. Indeed, for the linear operator-valued function $z\mathbf{G}_\alpha = \begin{bmatrix} z\mathbf{A} & z\mathbf{B} \\ z\mathbf{C} & z\mathbf{D} \end{bmatrix}$ corresponding to a conservative scattering system α we have proved in [5] that $z\mathbf{G}_\alpha \in S_N^0(\mathcal{X} \oplus \mathcal{U}, \mathcal{X} \oplus \mathcal{Y})$, hence for any N -tuple $\mathbf{T} = (T_1, \dots, T_N)$ of commuting contractions on some separable Hilbert space \mathcal{H} we obtain from (1.3) and (1.4)

$$\left\| \sum_{k=1}^N (G_\alpha)_k \otimes T_k \right\|_{[(\mathcal{X} \oplus \mathcal{U}) \otimes \mathcal{H}, (\mathcal{X} \oplus \mathcal{Y}) \otimes \mathcal{H}]} \leq 1$$

(for a finite sum in (1.4) one can pass to the limit in (1.3) as $r \uparrow 1$). Then

$$\left\| \sum_{k=1}^N B_k \otimes T_k \right\| = \left\| P_{\mathcal{X}} \otimes I_{\mathcal{H}} \left(\sum_{k=1}^N (G_\alpha)_k \otimes T_k \right) \right\|_{\mathcal{U} \otimes \mathcal{H}} \leq \left\| \sum_{k=1}^N (G_\alpha)_k \otimes T_k \right\| \leq 1$$

(here $P_{\mathcal{X}}$ is the orthoprojector onto \mathcal{X} in $\mathcal{X} \oplus \mathcal{Y}$), and by virtue of an arbitrariness of \mathcal{H} and \mathbf{T} we get $z\mathbf{B} \in S_N^0(\mathcal{U}, \mathcal{X})$. Analogously, $z\mathbf{A} \in S_N^0(\mathcal{X}, \mathcal{X})$, $z\mathbf{C} \in S_N^0(\mathcal{X}, \mathcal{Y})$, and by Corollary 2.3 the statement of Theorem 2.6 is valid for $m > 1$ also. \square

For the general case Problem 2.5 is still open. However we shall show how to reformulate this as the problem on the existence of a cascade decomposition for a conservative realization of $\theta(z)$ and give the criterion for its solvability in terms of common invariant subspaces for the N -tuple $\mathbf{A} = (A_1, \dots, A_N)$ of main operators of such a realization.

3 Cascade connections of multiparametric linear systems and factorizations of their transfer functions

In [4] we have introduced the notion of cascade connection of systems of the form (1.1) and established some of their properties. Recall that for systems $\alpha^{(1)} = (N; \mathbf{A}^{(1)}, \mathbf{B}^{(1)}, \mathbf{C}^{(1)}, \mathbf{D}^{(1)}; \mathcal{X}^{(1)}, \mathcal{U}, \mathcal{V})$ and $\alpha^{(2)} = (N; \mathbf{A}^{(2)}, \mathbf{B}^{(2)}, \mathbf{C}^{(2)}, \mathbf{D}^{(2)}; \mathcal{X}^{(2)}, \mathcal{V}, \mathcal{Y})$ their *cascade connection* is the system $\alpha = \alpha^{(2)}\alpha^{(1)} = (N; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}; \mathcal{X} = \mathcal{X}^{(2)} \oplus \mathcal{V} \oplus \mathcal{X}^{(1)}, \mathcal{U}, \mathcal{Y})$ where for any $z \in \mathbb{C}^N$

$$z\mathbf{G}_\alpha = \begin{bmatrix} z\mathbf{A} & z\mathbf{B} \\ z\mathbf{C} & z\mathbf{D} \end{bmatrix} := \begin{bmatrix} z\mathbf{A}^{(2)} & z\mathbf{B}^{(2)} & 0 & 0 \\ 0 & 0 & z\mathbf{C}^{(1)} & z\mathbf{D}^{(1)} \\ 0 & 0 & z\mathbf{A}^{(1)} & z\mathbf{B}^{(1)} \\ \hline z\mathbf{C}^{(2)} & z\mathbf{D}^{(2)} & 0 & 0 \end{bmatrix} \in [\mathcal{X}^{(2)} \oplus \mathcal{V} \oplus \mathcal{X}^{(1)} \oplus \mathcal{U}, \mathcal{X}^{(2)} \oplus \mathcal{V} \oplus \mathcal{X}^{(1)} \oplus \mathcal{Y}]. \quad (3.1)$$

Note that systems (1.1) have a unit delay, thus in contrast to the notion of cascade connection of systems without delay (see, e.g., [2] for the case $N = 1$) the state space \mathcal{X} of α contains an additional component—the intermediate space \mathcal{V} (see [4] for details). If both $\alpha^{(1)}$ and $\alpha^{(2)}$ are dissipative (resp., conservative) scattering systems then $\alpha = \alpha^{(2)}\alpha^{(1)}$ is also a dissipative (resp., conservative) scattering system. If $\theta_{\alpha^{(1)}}(z)$ and $\theta_{\alpha^{(2)}}(z)$ are holomorphic on some neighbourhood Γ of $z = 0$ then such is $\theta_\alpha(z)$ and

$$\theta_\alpha(z) = \theta_{\alpha^{(2)}\alpha^{(1)}}(z) = \theta_{\alpha^{(2)}}(z)\theta_{\alpha^{(1)}}(z). \quad (z \in \Gamma) \quad (3.2)$$

Theorem 3.1 *If $\alpha = \alpha^{(2)}\alpha^{(1)}$ is a closely connected system then both $\alpha^{(1)}$ and $\alpha^{(2)}$ are also closely connected.*

PROOF. It follows from (1.5) that \mathcal{X}_{cc} is the minimal subspace in \mathcal{X} containing $B_k\mathcal{U}$, $C_k^*\mathcal{Y}$ and reducing A_k for all $k \in \{1, \dots, N\}$. By the assumption, $\mathcal{X}_{cc} = \mathcal{X}$. If $\mathcal{X}_{cc}^{(1)} \neq \mathcal{X}^{(1)}$ then by (3.1) the subspace $\mathcal{X}^{(2)} \oplus \mathcal{V} \oplus \mathcal{X}_{cc}^{(1)} (\neq \mathcal{X})$ contains $B_k\mathcal{U}$, $C_k^*\mathcal{Y}$ and reduces A_k for all $k \in \{1, \dots, N\}$, that contradicts to the assumption. Hence, $\alpha^{(1)}$ is closely connected. Analogously, $\alpha^{(2)}$ is closely connected. \square

Note that the same is true for systems without delay for $N = 1$ (see [2]). The converse statement is false even for $N = 1$.

Example 3.2 Let $l^2 = \bigoplus_{n=-\infty}^{+\infty} \mathbb{C}$, $l_+^2 = \bigoplus_{n=0}^{+\infty} \mathbb{C}$, $l_-^2 = \bigoplus_{n=-\infty}^{-1} \mathbb{C}$ be Hilbert spaces of sequences. Clearly, $l^2 = l_+^2 \oplus l_-^2$. Let $U : l^2 \rightarrow l^2$ be the two-sided shift operator:

$$U : \text{col}(\dots, c_{-1}, \boxed{c_0}, c_1, \dots) \mapsto \text{col}(\dots, c_{-2}, \boxed{c_{-1}}, c_0, \dots).$$

Obviously, U is unitary. Define the systems $\alpha^{(j)} = (1; A^{(j)}, B^{(j)}, C^{(j)}, D^{(j)}; l_+^2, l_-^2, -l_-^2)$, $j = 1, 2$, where

$$\begin{bmatrix} A^{(1)} & B^{(1)} \\ C^{(1)} & D^{(1)} \end{bmatrix} = U^{-1} = U^* = \begin{bmatrix} A^{(2)} & B^{(2)} \\ C^{(2)} & D^{(2)} \end{bmatrix}^* \in [l_+^2 \oplus l_-^2, l_+^2 \oplus l_-^2].$$

In particular,

$$\begin{aligned} A^{(1)} : l_+^2 &\rightarrow l_+^2, & \text{col}(c_0, c_1, \dots) &\mapsto \text{col}(c_1, c_2, \dots), \\ C^{(1)} : l_+^2 &\rightarrow l_-^2, & \text{col}(c_0, c_1, \dots) &\mapsto \text{col}(\dots, 0, 0, c_0), \\ A^{(2)} : l_+^2 &\rightarrow l_+^2, & \text{col}(c_0, c_1, \dots) &\mapsto \text{col}(0, c_0, c_1, \dots), \\ B^{(2)} : l_-^2 &\rightarrow l_+^2, & \text{col}(\dots, c_{-2}, c_{-1}) &\mapsto \text{col}(c_{-1}, 0, 0, \dots). \end{aligned}$$

It is clear that both $\alpha^{(1)}$ and $\alpha^{(2)}$ are conservative scattering systems. Moreover, they are closely connected since $A^{(1)}$ and $A^{(2)}$ are respectively the backward shift and the forward shift operators on l_+^2 which are completely non-unitary. Let $\alpha = \alpha^{(2)}\alpha^{(1)}$. Then by (3.1) the main operator of α is

$$A = \begin{bmatrix} A^{(2)} & B^{(2)} & 0 \\ 0 & 0 & C^{(1)} \\ 0 & 0 & A^{(1)} \end{bmatrix} : l_+^2 \oplus l_-^2 \oplus l_+^2 \rightarrow l_+^2 \oplus l_-^2 \oplus l_+^2.$$

Let $\mathcal{K} := \{\text{col}(\dots, 0, 0, c_{-1}), c_{-1} \in \mathbb{C}\} \subset l_-^2$, and $\mathcal{X}_{\mathcal{K}} := l_+^2 \oplus \mathcal{K} \oplus l_+^2 \subset l_+^2 \oplus l_-^2 \oplus l_+^2$. Then A is acting on elements of $\mathcal{X}_{\mathcal{K}}$ as follows:

$$A : \begin{bmatrix} \text{col}(c_0^{(2)}, c_1^{(2)}, c_2^{(2)}, \dots) \\ \text{col}(\dots, 0, 0, c_{-1}) \\ \text{col}(c_0^{(1)}, c_1^{(1)}, c_2^{(1)}, \dots) \end{bmatrix} \mapsto \begin{bmatrix} \text{col}(c_{-1}, c_0^{(2)}, c_1^{(2)}, \dots) \\ \text{col}(\dots, 0, 0, c_0^{(1)}) \\ \text{col}(c_1^{(1)}, c_2^{(1)}, c_3^{(1)}, \dots) \end{bmatrix}.$$

It is clear now that $\mathcal{X}_{\mathcal{K}}$ is invariant subspace for A , and $A|\mathcal{X}_{\mathcal{K}}$ is unitary, thus operator A has a unitary part. Therefore the conservative scattering system α is not closely connected.

Note that the analogous (however, more complicated) example was constructed in [3] for the case of one-parametric conservative scattering systems without delay (in the language of unitary colligations).

Theorem 3.3 1. *Let $\alpha = (N; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a conservative scattering system, and $\mathcal{X}^{(2)}$ be a subspace of \mathcal{X} satisfying the following conditions:*

- (i) $\mathcal{X}^{(2)}$ is invariant for all A_k , $k = 1, \dots, N$ (equivalently, $\forall \zeta \in \mathbb{T}^N \zeta \mathbf{G}_{\alpha} \mathcal{X}^{(2)} \subset \mathcal{X}^{(2)} \oplus \mathcal{Y}$);
- (ii) the subspaces $\mathcal{V}_{\zeta} := (\zeta \mathbf{G}_{\alpha})^*(\mathcal{X}^{(2)} \oplus \mathcal{Y}) \ominus \mathcal{X}^{(2)}$ coincide for all $\zeta \in \mathbb{T}^N$ ($\forall \zeta \in \mathbb{T}^N \mathcal{V}_{\zeta} =: \mathcal{V}$).

Define $\mathcal{X}^{(1)} := \mathcal{X} \ominus (\mathcal{X}^{(2)} \oplus \mathcal{V})$, for all $k \in \{1, \dots, N\}$

$$\begin{aligned} (G_{\alpha^{(1)}})_k &:= P_{\mathcal{X}^{(1)} \oplus \mathcal{V}}(G_{\alpha})_k | \mathcal{X}^{(1)} \oplus \mathcal{U}, \\ (G_{\alpha^{(2)}})_k &:= P_{\mathcal{X}^{(2)} \oplus \mathcal{V}}(G_{\alpha})_k | \mathcal{X}^{(2)} \oplus \mathcal{V}, \end{aligned} \quad (3.3)$$

i.e.,

$$\begin{aligned} \begin{bmatrix} A_k^{(1)} & B_k^{(1)} \\ C_k^{(1)} & D_k^{(1)} \end{bmatrix} &:= \begin{bmatrix} P_{\mathcal{X}^{(1)} \oplus \mathcal{V}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} | \mathcal{X}^{(1)} \oplus \mathcal{U}, \\ \begin{bmatrix} A_k^{(2)} & B_k^{(2)} \\ C_k^{(2)} & D_k^{(2)} \end{bmatrix} &:= \begin{bmatrix} P_{\mathcal{X}^{(2)} \oplus \mathcal{V}} & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} | (\mathcal{X}^{(2)} \oplus \mathcal{V}) \oplus \{0\}. \end{aligned}$$

Then $\alpha = \alpha^{(2)}\alpha^{(1)}$ where $\alpha^{(1)} = (N; \mathbf{A}^{(1)}, \mathbf{B}^{(1)}, \mathbf{C}^{(1)}, \mathbf{D}^{(1)}; \mathcal{X}^{(1)}, \mathcal{U}, \mathcal{V})$, $\alpha^{(2)} = (N; \mathbf{A}^{(2)}, \mathbf{B}^{(2)}, \mathbf{C}^{(2)}, \mathbf{D}^{(2)}; \mathcal{X}^{(2)}, \mathcal{V}, \mathcal{Y})$; $\alpha^{(1)}$ and $\alpha^{(2)}$ are conservative scattering systems.

2. Any cascade connection of conservative scattering systems of the form (1.1) arises in this way.

PROOF. For any $\zeta \in \mathbb{T}^N$ $\zeta \mathbf{G}_{\alpha}$ is a unitary operator, and from (3.3) and the definition of \mathcal{V} we have $\zeta \mathbf{G}_{\alpha}(\mathcal{X}^{(2)} \oplus \mathcal{V}) = \mathcal{X}^{(2)} \oplus \mathcal{Y} = \zeta \mathbf{G}_{\alpha^{(2)}}(\mathcal{X}^{(2)} \oplus \mathcal{V})$, and $\zeta \mathbf{G}_{\alpha^{(2)}} = \zeta \mathbf{G}_{\alpha} | \mathcal{X}^{(2)} \oplus \mathcal{V}$ is an isometry, thus $\zeta \mathbf{G}_{\alpha^{(2)}}$ is unitary. Hence, $\alpha^{(2)}$ is a conservative scattering system. For any $\zeta \in \mathbb{T}^N$

$$\begin{aligned} \zeta \mathbf{G}_{\alpha}(\mathcal{X}^{(1)} \oplus \mathcal{U}) &= \zeta \mathbf{G}_{\alpha}(\mathcal{X} \oplus \mathcal{U}) \ominus \zeta \mathbf{G}_{\alpha}(\mathcal{X}^{(2)} \oplus \mathcal{V}) = (\mathcal{X} \oplus \mathcal{Y}) \ominus (\mathcal{X}^{(2)} \oplus \mathcal{Y}) \\ &= \mathcal{X} \ominus \mathcal{X}^{(2)} = \mathcal{X}^{(1)} \oplus \mathcal{V} = \zeta \mathbf{G}_{\alpha^{(1)}}(\mathcal{X}^{(1)} \oplus \mathcal{U}), \end{aligned}$$

and $\zeta \mathbf{G}_{\alpha^{(1)}} = \zeta \mathbf{G}_{\alpha} | \mathcal{X}^{(1)} \oplus \mathcal{U}$ is an isometry, thus $\zeta \mathbf{G}_{\alpha^{(1)}}$ is unitary. Hence, $\alpha^{(1)}$ is a conservative scattering system. It is easy to see now that for any $\zeta \in \mathbb{T}^N$ $\zeta \mathbf{G}_{\alpha}$ has a form (3.1), i.e. $\alpha = \alpha^{(2)}\alpha^{(1)}$. The second assertion of this theorem follows directly from the definition of cascade connection. \square

Note that Theorem 3.3 is an analogue of the well-known result for one-parametric systems without delay (Theorem 2.6 in [3]; see also Theorem 6.1 in [2]).

Theorem 3.4 For $\theta(z) \in S_N^0(\mathcal{U}, \mathcal{Y})$ such that $m(\theta) > 1$ Problem 2.5 is solvable if and only if there exists a closely connected conservative scattering system $\alpha_{cc} = (N; \mathbf{A}_{cc}, \mathbf{B}_{cc}, \mathbf{C}_{cc}, \mathbf{D}_{cc}; \mathcal{X}_{cc}, \mathcal{U}, \mathcal{Y})$ such that $\theta(z) = \theta_{\alpha_{cc}}(z)$ for $z \in \mathbb{D}^N$ and there exists a subspace $\mathcal{X}_{cc}^{(2)}$ in \mathcal{X}_{cc} satisfying conditions (i), (ii) in Theorem 3.3 applied for the system α_{cc} in the place of α .

PROOF. The part “if” is clear since in this case by Theorem 3.3 there are conservative scattering systems $\alpha^{(1)}$ and $\alpha^{(2)}$ such that $\alpha_{cc} = \alpha^{(2)}\alpha^{(1)}$, and by (3.2) we have

$\theta(z) = \theta_{\alpha_{cc}}(z) = \theta_{\alpha^{(2)}}(z)\theta_{\alpha^{(1)}}(z)$ with functions $\theta_{\alpha^{(1)}}(z)$ and $\theta_{\alpha^{(2)}}(z)$ belonging to the corresponding classes $S_N^0(\cdot, \cdot)$.

For the proof of the part “only if” let us assume that (2.5) holds with $\theta_1(z) \in S_N^0(\mathcal{U}, \mathcal{V})$, $\theta_2(z) \in S_N^0(\mathcal{V}, \mathcal{Y})$. Let $\alpha^{(1)} = (N; \mathbf{A}^{(1)}, \mathbf{B}^{(1)}, \mathbf{C}^{(1)}, \mathbf{D}^{(1)}; \mathcal{X}^{(1)}, \mathcal{U}, \mathcal{V})$, $\alpha^{(2)} = (N; \mathbf{A}^{(2)}, \mathbf{B}^{(2)}, \mathbf{C}^{(2)}, \mathbf{D}^{(2)}; \mathcal{X}^{(2)}, \mathcal{V}, \mathcal{Y})$ be some conservative realizations, respectively (which exist by [7]), i.e. $\theta_k(z) = \theta_{\alpha^{(k)}}(z)$, $k = 1, 2$. Then the conservative scattering system $\alpha = (N; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}; \mathcal{X}, \mathcal{U}, \mathcal{Y}) := \alpha^{(2)}\alpha^{(1)}$ has the transfer function $\theta_\alpha(z) = \theta_{\alpha^{(2)}}(z)\theta_{\alpha^{(1)}}(z)$, and the subspace $\mathcal{X}^{(2)}$ in \mathcal{X} satisfy conditions (i) and (ii) of Theorem 3.3. Define the subspace \mathcal{X}_{cc} in \mathcal{X} by (1.5), and operators $(A_{cc})_k := P_{\mathcal{X}_{cc}}A_k|\mathcal{X}_{cc}$, $(B_{cc})_k := P_{\mathcal{X}_{cc}}B_k$, $(C_{cc})_k := C_k|\mathcal{X}_{cc}$, $(D_{cc})_k := D_k = 0$, $k = 1, \dots, N$. Then, by Theorem 3.3 of [7], \mathcal{X}_{cc} is a reducing subspace in \mathcal{X} for all A_k , $k = 1, \dots, N$, and $\alpha_{cc} = (N; \mathbf{A}_{cc}, \mathbf{B}_{cc}, \mathbf{C}_{cc}, \mathbf{D}_{cc}; \mathcal{X}_{cc}, \mathcal{U}, \mathcal{Y})$ is a closely connected conservative realization of $\theta(z) = \theta_\alpha(z)$. Define $\mathcal{X}_{cc}^{(2)} := \overline{P_{\mathcal{X}_{cc}}\mathcal{X}^{(2)}}$. Then $\mathcal{X}_{cc}^{(2)}$ is an invariant subspace in \mathcal{X} for all A_k , $k = 1, \dots, N$. Since $(A_{cc})_k = A_k|\mathcal{X}_{cc}$ we obtain that $\mathcal{X}_{cc}^{(2)}$ is an invariant subspace in \mathcal{X}_{cc} for all $(A_{cc})_k$, $k = 1, \dots, N$, i.e. condition (i) in Theorem 3.3 is satisfied for $\mathcal{X}_{cc}^{(2)}$. For all $\zeta \in \mathbb{T}^N$ we have the spaces

$$\begin{aligned} (\mathcal{V}_{cc})_\zeta &:= (\zeta \mathbf{G}_{\alpha_{cc}})^*(\mathcal{X}_{cc}^{(2)} \oplus \mathcal{Y}) \ominus \mathcal{X}_{cc}^{(2)} \\ &= (\zeta \mathbf{G}_{\alpha_{cc}})^*(\overline{P_{\mathcal{X}_{cc}}\mathcal{X}^{(2)}} \oplus \mathcal{Y}) \ominus \overline{P_{\mathcal{X}_{cc}}\mathcal{X}^{(2)}} \\ &= (\zeta \mathbf{G}_\alpha)^* \overline{P_{\mathcal{X}_{cc} \oplus \mathcal{Y}}(\mathcal{X}^{(2)} \oplus \mathcal{Y})} \ominus \overline{P_{\mathcal{X}_{cc}}\mathcal{X}^{(2)}} \\ &= \overline{P_{(\zeta \mathbf{G}_\alpha)^*(\mathcal{X}_{cc} \oplus \mathcal{Y})}(\zeta \mathbf{G}_\alpha)^*(\mathcal{X}^{(2)} \oplus \mathcal{Y})} \ominus \overline{P_{\mathcal{X}_{cc}}\mathcal{X}^{(2)}} \\ &= \overline{P_{\mathcal{X}_{cc} \oplus \mathcal{U}}(\mathcal{X}^{(2)} \oplus \mathcal{V})} \ominus \overline{P_{\mathcal{X}_{cc}}\mathcal{X}^{(2)}} \\ &= \overline{P_{\mathcal{X}_{cc}}(\mathcal{X}^{(2)} \oplus \mathcal{V})} \ominus \overline{P_{\mathcal{X}_{cc}}\mathcal{X}^{(2)}} \quad (=: \mathcal{V}_{cc}) \end{aligned}$$

coinciding, and condition (ii) in Theorem 3.3 is also satisfied for $\mathcal{X}_{cc}^{(2)}$. The proof is complete. \square

Let us give some remarks. **1.** The subspace $\mathcal{X}_{cc}^{(2)}$ in Theorem 3.4 corresponds to the factorization of $\theta(z)$ which, in general, not necessarily coincides with the original factorization (2.5). **2.** The case when $\mathcal{X}_{cc}^{(2)} = \{0\}$ is also non-trivial (!) since for that $\mathcal{V}_{cc} = \overline{P_{\mathcal{X}_{cc}}\mathcal{V}} = (\zeta \mathbf{G}_{\alpha_{cc}})^*\mathcal{Y} = (\zeta \mathbf{G}_\alpha)^*\mathcal{Y} \neq \{0\}$, and the factor from the left in the corresponding factorization of $\theta(z)$ is a linear homogeneous operator-valued function. **3.** It is easy to show that one can define $\mathcal{X}_{cc}^{(2)}$ in another way, say, $\mathcal{X}_{cc}^{(2)} := \mathcal{X}_{cc} \cap \mathcal{X}^{(2)}$, and this subspace also satisfies conditions (i) and (ii) in Theorem 3.3.

References

[1] J. Agler, *On the representation of certain holomorphic functions defined on a polydisc*, Topics in Operator Theory: Ernst D. Hellinger Memorial Volume

(L. de Branges, I. Gohberg, and J. Rovnyak, eds.), Oper. Theory and Appl., vol. 48, Birkhäuser-Verlag, Basel, 1990, pp. 47–66.

- [2] J.A. Ball and N. Cohen, *De Branges-Rovnyak operator models and systems theory: a survey*, Topics in Matrix and Operator Theory (H. Bart, I. Gohberg, and M.A. Kaashoek, eds.), Oper. Theory Adv. Appl., vol. 50, Birkhäuser-Verlag, Basel, 1991, pp. 93–136.
- [3] M.S. Brodskii, *Unitary operator colligations and their characteristic functions*, Uspekhi Mat. Nauk **33** (1978), no. 4, 141–168, (Russian).
- [4] D. S. Kalyuzhniy, *Cascade connections of multiparametric linear systems and the conservative realization of a decomposable inner operator function on the bidisk*, Mat. Studii, to appear (Russian).
- [5] ———, *Multiparametric dissipative linear stationary dynamical scattering systems: Discrete case, II: Existence of conservative dilations*, Integral Equations Operator Theory, to appear; e-print archive FA/9810119.
- [6] ———, *On the notions of dilation, controllability, observability, and minimality in the theory of dissipative scattering linear nD systems*, Preprint.
- [7] ———, *Multiparametric dissipative linear stationary dynamical scattering systems: Discrete case*, J. Operator Theory **44** (2000), to appear.
- [8] W. Rudin, *Function theory in polydiscs*, Benjamin, New York, 1969.
- [9] J. von Neumann, *Eine Spectraltheorie für allgemeine Operatoren eines unitären Raumes*, Math. Nachr. **4** (1951), 258–281.

Department of Higher Mathematics
 Odessa State Academy of Civil Engineering and Architecture
 Didrihson str. 4, Odessa, 270029, Ukraine